SOME PROPERTIES OF FRÉCHET MEDIANS IN RIEMANNIAN MANIFOLDS

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ABSTRACT. The consistency of Fréchet medians is proved for probability measures in proper metric spaces. In the context of Riemannian manifolds, assuming that the probability measure has more than a half mass lying in a convex ball and verifies some concentration conditions, the positions of its Fréchet medians are estimated. It is also shown that, in compact Riemannian manifolds, the Fréchet sample medians of generic data points are always unique.

1. INTRODUCTION

The history of medians can be dated back to 1629 when P. Fermat initiated a challenge (see [9]): given three points in the plan, find a fourth one such that the sum of its distances to the three given points is minimum. The answer to this question, which was firstly found by E. Torricelli (see [17]) in 1647, is that if each angle of the triangle is smaller than $2\pi/3$, then the minimum point is such that the three segments joining it and the vertices of the triangle form three angles equal to $2\pi/3$; and in the opposite case, the minimum point is the vertex whose angle is greater than or equal to $2\pi/3$. This point is called the median or the Fermat point of the triangle.

The notion of median also appears in statistics since a long time ago. In 1774, when P. S. Laplace tried to find an appropriate notion of the middle point for a group of observation values, he introduced in [13] "the middle of probability", the point that minimizes the sum of its absolute differences to data points, this is exactly the one dimensional median.

A sufficiently general notion of median in metric spaces is proposed in 1948 by M. Fréchet in his famous article [10], where he defines a p-mean of a random variable X to be a point which minimizes the expectation of its distance at the power p to X. In practice, two important cases are $p = 1$ and $p = 2$, which correspond to the notions of Fréchet median and Fréchet mean, respectively. Probably the most significant advantage of the median over the mean is that the former is robust but the latter is not, that is to say, the median is much less sensitive to outliers than the mean. Roughly speaking (see [15]), in order to move the median of a group of data points to arbitrarily far, at least a half of data points should be moved. On the contrary, in order to move the mean of a group of data points to arbitrarily far, it suffices to move one data point. So that medians are in some sense more prudent than means, as argued by M. Fréchet. The robustness property makes the

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median an important estimator in situations when there are lots of noise and disturbing factors.

Under the framework of Riemannian manifolds, the existence and uniqueness of local medians are proved in [19] for probability measures whose support are contained in a convex geodesic ball. It should be noted that, if the local curvature conditions in [19] are replaced by global ones, then it is shown in [1] that the local medians are in fact global medians, that is, Fréchet medians. Stochastic algorithms and deterministic algorithms for computing medians can be found in [3] and [19].

The aim of this paper is to give some basic properties of Fréchet medians. Firstly, we consider the question of consistency, which is important on the estimation of location. Theorem 2.3 states that, in proper metric spaces, if the first moment functions of a sequence of probability measures converge uniformly to the first moment function of another probability measure, then the corresponding sequence of Fréchet median sets also converges to the Fréchet median sets of the limiting measure. As a result, if a probability measure has only one Fréchet median, then any sequence of empirical Fréchet medians will converge almost surely to it. In the second section, we study the robustness of Fréchet medians in Riemannian manifolds. In Euclidean spaces, it is shown in [15] that if a group of data points has more than a half concentrated in a bounded region, then its Fréchet median cannot be drown arbitrarily far when the other points move. A generalization and refinement of this result for data points in Riemannian manifolds is given in Theorem 3.14, where an upper bound of the furthest distance to which Fréchet medians can move is given in terms of the upper bound of sectional curvatures, the concentration radius and the concentrated mass of the probability measure. This theorem also generalizes a result in [1] which states that if the probability measure is supported in a strongly convex ball, then all its Fréchet medians lie in that ball. Moreover, though we have chosen the framework of studying the robustness of Fréchet medians to be Riemannian manifolds, it is easily seen that our results remain true for general CAT(Δ) spaces. Finally, the uniqueness question of Fréchet sample medians is considered in the context of compact Riemannian manifolds. It is shown that, apart from several events of probability zero, the Fréchet sample medians are unique if the sample vector has a density with respect to the canonical Lebesgue measure of the product manifold. In other words, the Fréchet medians of generic data points are always unique.

2. Consistency of Fréchet medians in metric spaces

Let (M, d) be a proper metric space (recall that a metric space is proper if and only if every bounded and closed subset is compact) and $P_1(M)$ denote the set of all the probability measures μ on M verifying

$$\int_M d(x_0, p)\mu(dp) < \infty, \text{ for some } x_0 \in M.$$

For every $\mu \in P_1(M)$ we can define a function

$$f_\mu : \qquad M \longrightarrow \mathbf{R}_+ , \qquad x \longmapsto \int_M d(x, p)\mu(dp).$$

This function is 1-Lipschitz hence continuous on M. Since M is proper, f_μ attains its minimum (see [16, p. 42]), so we can give the following definition:

Definition 2.1. Let μ be a probability measure in $P_1(M)$, then a *global* minimum point of f_μ is called a Fréchet median of μ. The set of all the Fréchet medians of μ is denoted by Q_μ. Let f_μ^* denote the global minimum of f_μ.

Observe that Q_μ is compact, since the triangle inequality implies that $d(x, y) \leq 2f_\mu^*$ for every $x, y \in Q_\mu$.

To introduce the next proposition, let us recall that the L^1-Wasserstein distance between two elements μ and ν in $P_1(M)$ is defined by

$$W_1(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int_{M \times M} d(x, y) d\pi(x, y),$$

where $\Pi(\mu, \nu)$ is the set of all the probability measures on $M \times M$ with margins μ and ν. As a useful case for us, observe that $f_\mu(x) = W_1(\delta_x, \mu)$ for every $x \in M$. The set of all the 1-Lipschitz functions on M is denoted by $\mathrm{Lip}_1(M)$.

As is well known that Riemannian barycenters are characterized by convex functions (see [12, Lemma 7.2]), the following proposition shows that Fréchet medians can be characterized by Lipschitz functions.

Proposition 2.2. *Let $\mu \in P_1(M)$ and M be also separable, then*

$$Q_\mu = \left\{ x \in M : \varphi(x) \leq f_\mu^* + \int_M \varphi(p)\mu(dp), \text{ for every } \varphi \in \mathrm{Lip}_1(M) \right\}.$$

Proof. The separability of M ensures that the duality formula of Kantorovich-Rubinstein (see [18, p. 107]) can be applied, so that for every $x \in M$,

$$\begin{aligned} x \in Q_\mu &\Longleftrightarrow f_\mu(x) \leq f_\mu^* \\ &\Longleftrightarrow W_1(\delta_x, \mu) \leq f_\mu^* \\ &\Longleftrightarrow \sup_{\varphi \in \mathrm{Lip}_1(M)} \left| \varphi(x) - \int_M \varphi(p)\mu(dp) \right| \leq f_\mu^* \\ &\Longleftrightarrow \varphi(x) \leq f_\mu^* + \int_M \varphi(p)\mu(dp), \text{ for every } \varphi \in \mathrm{Lip}_1(M), \end{aligned}$$

as desired. □

We proceed to show the main result of this section.

Theorem 2.3. *Let $(\mu_n)_{n \in \mathbf{N}}$ be a sequence in $P_1(M)$ and μ be another probability measure in $P_1(M)$. If $(f_{\mu_n})_n$ converges uniformly on M to f_μ, then for every $\varepsilon > 0$, there exists $N \in \mathbf{N}$, such that for every $n \geq N$ we have*

$$Q_{\mu_n} \subset B(Q_\mu, \varepsilon) := \{x \in M : d(x, Q_\mu) < \varepsilon\}.$$

Proof. We prove this by contradiction. Suppose that the assertion is not true, then without loss of generality, we can assume that there exist some $\varepsilon > 0$ and a sequence $(x_n)_{n \in \mathbf{N}}$ such that $x_n \in Q_{\mu_n}$ and $x_n \notin B(Q_\mu, \varepsilon)$ for every n. If this sequence is bounded, then by choosing a subsequence we can assume that $(x_n)_n$ converges to a point $x_* \notin B(Q_\mu, \varepsilon)$ because $M \setminus B(Q_\mu, \varepsilon)$

is closed. However, observe that the uniform convergence of $(f_{\mu_n})_n$ to f_μ implies $f^*_{\mu_n} \longrightarrow f^*_\mu$, hence one gets

$$\begin{aligned}|f_\mu(x_*) - f^*_\mu| &\leq |f_\mu(x_*) - f_{\mu_n}(x_*)| + |f_{\mu_n}(x_*) - f_{\mu_n}(x_n)| + |f_{\mu_n}(x_n) - f^*_\mu| \\ &\leq \sup_{x\in M} |f_{\mu_n}(x) - f_\mu(x)| + d(x_*, x_n) + |f^*_{\mu_n} - f^*_\mu| \longrightarrow 0.\end{aligned}$$

So that $f_\mu(x_*) = f^*_\mu$, that is to say $x_* \in Q_\mu$. This is impossible, hence $(x_n)_n$ is not bounded. Now we fix a point $\bar{x} \in Q_\mu$, always by choosing a subsequence we can assume that $d(x_n, \bar{x}) \longrightarrow +\infty$, then

$$\begin{aligned}f_\mu(x_n) &= \int_M d(x_n, p)\mu(dp) \geq \int_M (d(x_n, \bar{x}) - d(\bar{x}, p))\mu(dp) \\ &= d(x_n, \bar{x}) - f^*_\mu \longrightarrow +\infty. \end{aligned} \tag{1}$$

On the other hand,

$$\begin{aligned}|f_\mu(x_n) - f^*_\mu| &\leq |f_\mu(x_n) - f_{\mu_n}(x_n)| + |f_{\mu_n}(x_n) - f^*_\mu| \\ &\leq \sup_{x\in M} |f_{\mu_n}(x) - f_\mu(x)| + |f^*_{\mu_n} - f^*_\mu| \longrightarrow 0.\end{aligned}$$

This contradicts (1), the proof is complete. □

Remark 2.4. A sufficient condition to ensure the uniform convergence of $(f_{\mu_n})_n$ on M to f_μ is that $W_1(\mu_n, \mu) \longrightarrow 0$, since

$$\sup_{x\in M} |f_{\mu_n}(x) - f_\mu(x)| = \sup_{x\in M} |W_1(\delta_x, \mu_n) - W_1(\delta_x, \mu)| \leq W_1(\mu_n, \mu).$$

The consistency of Fréchet means is proved in [5, Theorem 2.3]. The consistency of Fréchet medians given below is a corollary to Theorem 2.3. A similar result can be found in [16, p. 44].

Corollary 2.5. *Let $(X_n)_{n\in\mathbf{N}}$ be a sequence of i.i.d random variables of law $\mu \in P_1(M)$ and $(m_n)_{n\in\mathbf{N}}$ be a sequence of random variables such that $m_n \in Q_{\mu_n}$ with $\mu_n = \frac{1}{n}\sum_{k=1}^n \delta_{X_k}$. If μ has a unique Fréchet median m, then $m_n \longrightarrow m$ a.s.*

Proof. By Theorem 2.3 and Remark 2.4, it suffices to show that $\mu_n \xrightarrow{W_1} \mu$ a.s. This is equivalent to show that (see [18, p. 108]) for every $f \in C_b(M)$,

$$\frac{1}{n}\sum_{k=1}^n f(X_k) \longrightarrow \int_M f(p)\mu(dp) \quad \text{a.s.}$$

and for every $x \in M$,

$$\frac{1}{n}\sum_{k=1}^n d(x, X_k) \longrightarrow \int_M d(x, p)\mu(dp) \quad \text{a.s.}$$

These two assertions are trivial corollaries to the strong law of large numbers, hence the result holds. □

3. ROBUSTNESS OF FRÉCHET MEDIANS IN RIEMANNIAN MANIFOLDS

Throughout this section, we assume that M is a complete Riemannian manifold with dimension no less than 2, whose Riemannian distance is denoted by d. We fix a closed geodesic ball

$$\bar{B}(a,\rho)=\{x\in M: d(x,a)\le\rho\}$$

in M centered at a with a finite radius $\rho>0$ and a probability measure $\mu\in P_1(M)$ such that

$$\mu(\bar{B}(a,\rho))=\alpha>\frac{1}{2}.$$

The aim of this section is to estimate the positions of the Fréchet medians of μ, which gives a quantitative estimation for robustness. To this end, the following type of functions are of fundamental importance for our methods. Let $x,z\in M$, define

$$h_{x,z}:\quad \bar{B}(a,\rho)\longrightarrow\mathbf{R},\quad p\longmapsto d(x,p)-d(z,p).$$

Obviously, $h_{x,z}$ is continuous and attains its minimum.

Our method of estimating the position of Q_μ is essentially based on the following simple observation.

Proposition 3.1. *Let $x\in\bar{B}(a,\rho)^c$ and assume that there exists $z\in M$ such that*

$$\min_{p\in\bar{B}(a,\rho)} h_{x,z}(p)>\frac{1-\alpha}{\alpha}\,d(x,z),$$

then $x\notin Q_\mu$.

Proof. Clearly one has

$$\begin{aligned}f_\mu(x)-f_\mu(z)&=\int_{\bar{B}(a,\rho)}(d(x,p)-d(z,p))\mu(dp)+\int_{M\setminus\bar{B}(a,\rho)}(d(x,p)-d(z,p))\mu(dp)\\&\ge\alpha\min_{p\in\bar{B}(a,\rho)}h_{x,z}(p)-(1-\alpha)d(x,z)>0.\end{aligned}$$

The proof is complete. □

By choosing the dominating point $z=a$ in Proposition 3.1 we get the following basic estimation.

Theorem 3.2. *The set Q_μ of all the Fréchet medians of μ verifies*

$$Q_\mu\subset\bar{B}\Big(a,\frac{2\alpha\rho}{2\alpha-1}\Big).$$

Proof. Observe that for every $p\in\bar{B}(a,\rho)$,

$$h_{x,a}(p)=d(x,p)-d(a,p)\ge d(x,a)-2d(a,p)\ge d(x,a)-2\rho.$$

Hence Proposition 3.1 yields

$$\begin{aligned}Q_\mu\cap\bar{B}(a,\rho)^c&\subset\big\{x\in M:\min_{p\in\bar{B}(a,\rho)}h_{x,a}(p)\le\frac{1-\alpha}{\alpha}\,d(x,a)\big\}\\&\subset\big\{x\in M: d(x,a)-2\rho\le\frac{1-\alpha}{\alpha}\,d(x,a)\big\}\\&=\big\{x\in M: d(x,a)\le\frac{2\alpha\rho}{2\alpha-1}\big\}.\end{aligned}$$

The proof is complete. □

Remark 3.3. It is easily seen that the conclusions of Proposition 3.1 and Theorem 3.2 also hold if M is only a proper metric space.

Remark 3.4. As a direct corollary to Theorem 3.2, if μ is a probability measure in $P_1(M)$ such that for some point $m \in M$ one has $\mu\{m\} > 1/2$, then m is the unique Fréchet median of μ.

Thanks to Theorem 3.2, from now on we only have to work in the closed geodesic ball

$$B_* = \bar{B}\Big(a, \frac{2\alpha\rho}{2\alpha - 1}\Big).$$

Thus let Δ be an upper bound of sectional curvatures in B_* and inj be the injectivity radius of B_*. Moreover, we shall always assume that the following concentration condition is fulfilled throughout the rest part of this section:

Assumption 3.5.

$$\frac{2\alpha\rho}{2\alpha - 1} < r_* := \min\{\frac{\pi}{\sqrt{\Delta}}, \mathrm{inj}\},$$

where if $\Delta \leq 0$, then $\pi/\sqrt{\Delta}$ is interpreted as $+\infty$.

In view of Proposition 3.1 and Theorem 3.2, estimating the position of Q_μ can be achieved by estimating the minimum of the functions $h_{x,z}$ for some $x, z \in B_*$. The following lemma enables us to use the comparison argument proposed in [1] to compare the configurations in B_* with the ones in model spaces in order to obtain lower bounds of the functions $h_{x,z}$.

Lemma 3.6. *Let $x \in B_* \setminus \bar{B}(a,\rho)$ and y be the intersection point of the boundary of $\bar{B}(a,\rho)$ and the minimal geodesic joining x and a. Let $z \neq x$ be another point on the minimal geodesic joining x and a. Assume that $d(a,x) + d(a,z) < r_*$, then*

$$\operatorname{argmin} h_{x,z} \subset \{p \in \bar{B}(a,\rho) : \ d(x,p) + d(p,z) + d(z,x) < 2r_*\}.$$

Proof. Let $p \in \bar{B}(a,\rho)$ such that $d(x,p) + d(p,z) + d(z,x) \geq 2r_*$, then

$$\begin{aligned} h_{x,z}(p) &\geq 2r_* - d(x,z) - 2d(z,p) \\ &> 2(d(a,x) + d(a,z)) - d(x,z) - 2(d(a,z) + \rho) \\ &= d(x,y) - d(a,y) + d(a,z). \end{aligned} \tag{2}$$

If $d(a,y) > d(a,z)$, then (2) yields $h_{x,z}(p) > h_{x,z}(y)$, thus p cannot be a minimum point of $h_{x,z}$. On the other hand, if $d(a,y) \leq d(a,z)$, then (2) gives that $h_{x,z}(p) > d(x,y) + d(y,z) \geq d(x,z)$, which is impossible. Hence in either case, every minimum point p of $h_{x,z}$ must verify $d(x,p) + d(p,z) + d(z,x) < 2r_*$. □

As a preparation for the comparison arguments in the following, let us recall the definition of model spaces. For a real number κ, the model space $\mathbb{M}^2_\kappa$ is defined as follows:
1) if $\kappa > 0$, then $\mathbb{M}^2_\kappa$ is obtained from the sphere $\mathbb{S}^2$ by multiplying the distance function by $1/\sqrt{\kappa}$;
2) if $\kappa = 0$, then $\mathbb{M}^2_\kappa$ is the Euclidean space $\mathbb{E}^2$;

3) if $\kappa < 0$, then $\mathbb{M}_\kappa^2$ is obtained from the hyperbolic space $\mathbb{H}^2$ by multiplying the distance function by $1/\sqrt{-\kappa}$.
Moreover, the distance between two points $\bar{x}$ and $\bar{y}$ in $\mathbb{M}_\kappa^2$ will be denoted by $\bar{d}(\bar{x}, \bar{y})$.

The following proposition says that for the positions of Fréchet medians, if comparisons can be done, then the model space $\mathbb{M}_\Delta^2$ is the worst case.

Proposition 3.7. *Consider in $\mathbb{M}_\Delta^2$ the same configuration as that in Lemma 3.6: a closed geodesic ball $\bar{B}(\bar{a}, \rho)$ and a point $\bar{x}$ such that $\bar{d}(\bar{x}, \bar{a}) = d(x, a)$. We denote $\bar{y}$ the intersection point of the boundary of $\bar{B}(\bar{a}, \rho)$ and the minimal geodesic joining $\bar{x}$ and $\bar{a}$. Let $\bar{z}$ be a point in the minimal geodesic joining $\bar{x}$ and $\bar{a}$ such that $\bar{d}(\bar{a}, \bar{z}) = d(a, z)$. Assume that $d(a, x) + d(a, z) < r_*$, then*

$$\min_{p \in \bar{B}(a,\rho)} h_{x,z}(p) \geq \min_{\bar{p} \in \bar{B}(\bar{a},\rho)} \bar{h}_{\bar{x},\bar{z}}(\bar{p}),$$

where $\bar{h}_{\bar{x},\bar{z}}(\bar{p}) := \bar{d}(\bar{x}, \bar{p}) - \bar{d}(\bar{z}, \bar{p})$.

Proof. Let $p \in \operatorname{argmin} h_{x,z}$. Consider a comparison point $\bar{p} \in \mathbb{M}_\Delta^2$ such that $\bar{d}(\bar{z}, \bar{p}) = d(z, p)$ and $\angle \bar{a}\bar{z}\bar{p} = \angle azp$. Then the assumption $d(a, x) + d(a, z) < r_*$ and the hinge version of Alexandrov-Toponogov comparison theorem (see [7, Exercise IX.1, p. 420]) yield that $\bar{d}(\bar{a}, \bar{p}) \leq d(a, p) = \rho$, i.e. $\bar{p} \in \bar{B}(\bar{a}, \rho)$. Now by hinge comparison again and Lemma 3.6, we get $\bar{d}(\bar{p}, \bar{x}) \leq d(p, x)$, which implies that

$$h_{x,z}(p) \geq \bar{h}_{\bar{x},\bar{z}}(\bar{p}) \geq \min_{\bar{p} \in \bar{B}(\bar{a},\rho)} \bar{h}_{\bar{x},\bar{z}}(\bar{p}).$$

The proof is complete. $\square$

According to Proposition 3.7, it suffices to find the minima of the functions $h_{x,z}$ when M equals $\mathbb{S}^2$, $\mathbb{E}^2$ and $\mathbb{H}^2$, which are of constant curvatures 1, 0 and -1, respectively.

Proposition 3.8. *Let $t, u \geq 0$ such that $u < \rho + t \leq 2\alpha\rho/(2\alpha - 1)$.*

i) If $M = \mathbb{S}^2$, let $x = (\sin(\rho + t), 0, \cos(\rho + t))$ and $z = (\sin u, 0, \cos u)$. Assume that $\rho + t + u < \pi$, then

$$\min_{\bar{B}(a,\rho)} h_{x,z} = \begin{cases} t - \rho + u, & \text{if} \quad \cot u \geq 2\cot\rho - \cot(\rho + t); \\ \arccos\left(\cos(\rho + t - u) + \dfrac{\sin^2\rho \sin^2(\rho + t - u)}{2\sin u \sin(\rho + t)}\right), & \text{if} \quad \text{not}. \end{cases}$$

ii) If $M = \mathbb{E}^2$, let $a = (0, 0)$, $x = (\rho + t, 0)$, $z = (u, 0)$, then

$$\min_{\bar{B}(a,\rho)} h_{x,z} = \begin{cases} t - \rho + u, & \text{if} \quad u \leq \dfrac{(\rho + t)\rho}{\rho + 2t}; \\ (\rho + t - u)\sqrt{1 - \dfrac{\rho^2}{u(\rho + t)}}, & \text{if} \quad \text{not}. \end{cases}$$

iii) If $M = \mathbb{H}^2$, let $a = (0,0,1)$, $x = (\sinh(\rho+t), 0, \cosh(\rho+t))$ and $z = (\sinh u, 0, \cosh u)$, then

$$\min_{\bar{B}(a,\rho)} h_{x,z} = \begin{cases} t-\rho+u, & \textit{if} \quad \coth u \geq 2\coth\rho - \coth(\rho+t); \\ \operatorname{arccosh}\Big(\cosh(\rho+t-u) - \dfrac{\sinh^2\rho\sinh^2(\rho+t-u)}{2\sinh u\sinh(\rho+t)}\Big), & \textit{if} \quad \textit{not.} \end{cases}$$

We shall only prove the result for the case when $M = \mathbb{S}^2$, since the proofs for $M = \mathbb{E}^2$ and $M = \mathbb{H}^2$ are similar and easier. The proof consists of some lemmas, the first one below says that $h_{x,z}$ is smooth at its minimum points which can only appear on the boundary of the ball $\bar{B}(a,\rho)$.

Lemma 3.9. *Let x' and z' be the antipodes of x and z. Then $z' \notin \bar{B}(a,\rho)$ and all the local minimum points of $h_{x,z}$ are contained in $\partial\bar{B}(a,\rho) \setminus \{x'\}$.*

Proof. It is easily seen that $d(z', a) = \pi - u > \rho$, so that $z' \notin \bar{B}(a,\rho)$. Observe that x' is a global maximum point of $h_{x,z}$ which is not locally constant, so that x' cannot be a local minimum. Now let $p \in B(a,\rho)$ be a local minimum of $h_{x,z}$, then $h_{x,z}$ is smooth at p. It follows that $\operatorname{grad} h_{x,z}(p) = 0$, which yields that $h_{x,z}(p) = d(x,z)$, this is a contradiction. The proof is complete. ☐

The following lemma characterizes the global minimum points of $h_{x,z}$.

Lemma 3.10. *The set of global minimum points of $h_{x,z}$ verifies*

$$\operatorname{argmin} h_{x,z} = \begin{cases} \{y\}, & \textit{if} \cot u \geq 2\cot\rho - \cot(\rho+t); \\ \{\, p \in \partial\bar{B}(a,\rho) : \dfrac{\sin(\rho+t)}{\sin d(x,p)} = \dfrac{\sin u}{\sin d(z,p)}\}, & \textit{if not,} \end{cases}$$

where y is the intersection point of the boundary of $\bar{B}(a,\rho)$ and the minimal geodesic joining x and a.

Proof. Thanks to Lemma 3.9, it suffices to find the global minimum points of $h_{x,z}$ for $p = (\sin\rho\cos\theta, \sin\rho\sin\theta, \cos\rho)$ and $\theta \in [0, 2\pi)$. In this case,

$$\begin{aligned} h_{x,z}(p) &= d(x,p) - d(z,p) \\ &= \arccos(\sin(\rho+t)\sin\rho\cos\theta + \cos(\rho+t)\cos\rho) \\ &\quad - \arccos(\sin u\sin\rho\cos\theta + \cos u\cos\rho) \\ &:= h(\theta). \end{aligned}$$

Hence let $p = (\sin\rho\cos\theta, \sin\rho\sin\theta, \cos\rho)$ be a local minimum point of $h_{x,z}$, then Lemma 3.9 yields that $h'(\theta)$ exists and equals zero. On the other hand, by elementary calculation,

$$\begin{aligned} h'(\theta) &= \sin\rho\sin\theta\Big(\frac{\sin(\rho+t)}{\sqrt{1-(\sin(\rho+t)\sin\rho\cos\theta+\cos(\rho+t)\cos\rho)^2}} \\ &\qquad - \frac{\sin u}{\sqrt{1-(\sin u\sin\rho\cos\theta+\cos u\cos\rho)^2}}\Big) \\ &= \sin\rho\sin\theta\Big(\frac{\sin(\rho+t)}{\sin d(x,p)} - \frac{\sin u}{\sin d(z,p)}\Big). \end{aligned}$$

Hence we have necessarily

$$\theta = 0,\ \pi \quad \text{or} \quad \frac{\sin(\rho+t)}{\sin d(x,p)} = \frac{\sin u}{\sin d(z,p)}.$$

Firstly, we observe that w, the corresponding point p when $\theta = \pi$, cannot be a minimum point. In fact, let w' be the antipode of w. If $d(x,a) < d(w',a)$, then $h_{x,z}(w) = d(x,z)$. So that w is a maximum point. On the other hand, if $d(x,a) \geq d(w',a)$, then $d(w,x)+d(x,z)+d(z,w) \equiv 2\pi$. Hence Lemma 3.6 and the condition $\rho+t+u<\pi$ imply that w is not a minimum point. So that the assertion holds.

Now assume that $p \neq w, y$ such that

$$\frac{\sin(\rho+t)}{\sin d(x,p)} = \frac{\sin u}{\sin d(z,p)}. \tag{3}$$

Let $\beta = \angle zpa$. Then by the spherical law of sines, (3) is equivalent to $\sin(\beta + \angle zpx) = \sin\beta$, i.e. that $\angle zpx = \pi - 2\beta$. Applying the spherical law of sines to $\triangle zpx$ we get

$$\frac{\sin\angle x}{\sin d(z,p)} = \frac{\sin 2\beta}{\sin(\rho+t-u)}. \tag{4}$$

By the spherical law of sines in $\triangle apx$,

$$\frac{\sin\angle x}{\sin\rho} = \frac{\sin\beta}{\sin(\rho+t)}. \tag{5}$$

Then (4)/(5) gives that

$$\sin d(z,p)\cos\beta = \frac{\sin\rho\sin(\rho+t-u)}{2\sin(\rho+t)}. \tag{6}$$

By the spherical law of cosines in $\triangle azp$,

$$\sin d(z,p)\cos\beta = \frac{\cos u - \cos\rho\cos d(z,p)}{\sin\rho}. \tag{7}$$

Then (6) and (7) give that

$$\cos d(z,p) = \frac{2\cos u\sin(\rho+t) - \sin^2\rho\sin(\rho+t-u)}{2\cos\rho\sin(\rho+t)}. \tag{8}$$

Moreover, by (8) and spherical law of cosines in $\triangle azp$,

$$\cos\theta = \frac{\tan\rho}{2}(\cot u + \cot(\rho+t)). \tag{9}$$

The condition $0<\rho+t+u<\pi$ and (9) give that

$$\cos\theta = \frac{\tan\rho\sin(\rho+t+u)}{2\sin u\sin(\rho+t)} > 0. \tag{10}$$

Furthermore, considering $p \neq y$ we must have $\cos\theta < 1$. By (9) this is equivalent to

$$\cot u < 2\cot\rho - \cot(\rho+t), \tag{11}$$

which is also equivalent to

$$\frac{\sin(\rho-u)}{\sin u} < \frac{\sin t}{\sin(\rho+t)}. \tag{12}$$

For the case when $u \geq \rho$ it is easily seen that y is a maximum point of $h_{x,z}$ and hence p must verify (3) and the corresponding θ is determined by (9). Hence in this case, there are exactly two local minimum points of $h_{x,z}$ and obviously they are also global ones.

Now let $u < \rho$, then easy computation gives

$$h''(0) = \sin\rho\Big(\frac{\sin(\rho+t)}{\sin t} - \frac{\sin u}{\sin(\rho-u)}\Big). \tag{13}$$

Hence if $\frac{\sin(\rho+t)}{\sin t} \geq \frac{\sin u}{\sin(\rho-u)}$, then (12) yields that y is the unique global minimum of $h_{x,z}$. In the opposite case, (13) implies that y is a local maximum point. Hence the same argument as in the case when $u \geq \rho$ completes the proof of lemma. □

We need the following technical lemma.

Lemma 3.11. *If* $\cot u < 2\cot\rho - \cot(\rho+t)$*, then* $0 < d(x,p) - d(z,p) < \pi$ *for every* $p \in \operatorname{argmin} h_{x,z}$*.*

Proof. It suffices to show $d(x,p) > d(z,p)$. For the case when $u < \rho$, we firstly show that $d(z,p) < \pi/2$. In fact, by (8) this is equivalent to show that

$$(1+\cos^2\rho)\cos u \sin(\rho+t) + \sin^2\rho \sin u \cos(\rho+t) > 0 \tag{14}$$

If $\rho + t \leq \pi/2$, (14) is trivially true. Now assume $\rho + t > \pi/2$. So that $\pi/2 < \rho+t < \pi-u$, which implies $\sin(\rho+t) > \sin u$ and $\cos(\rho+t) > -\cos u$. Hence we get

$$\begin{aligned}&(1+\cos^2\rho)\cos u \sin(\rho+t) + \sin^2\rho \sin u \cos(\rho+t)\\ >&(1+\cos^2\rho)\cos u \sin u - \sin^2\rho \sin u \cos u\\ =&2\cos^2\rho \cos u \sin u > 0.\end{aligned}$$

So that $d(z,p) < \pi/2$ holds. Now if $d(x,p) \geq \pi/2$, then obviously $d(x,p) > d(z,p)$. So that assume $d(x,p) < \pi/2$. Observe that $\rho + t + u < \pi$ implies $\sin u < \sin(\rho+t)$, then (3) yields $d(x,p) > d(z,p)$.

For the case when $u \geq \rho$, it suffices to show that $\cos d(z,p) > \cos d(x,p)$ for every $p = (\sin\rho\cos\theta, \sin\rho\sin\theta, \cos\rho)$ with $\theta \in [0,\pi]$. Now let

$$\begin{aligned}g(\theta) &= \sin\rho\cos\theta(\sin u - \sin(\rho+t)) + \cos\rho(\cos u - \cos(\rho+t))\\ &= \cos d(z,p) - \cos d(x,p)\end{aligned}$$

Then $g'(\theta) = -\sin\rho\sin\theta(\sin u - \sin(\rho+t))$. Observe that $\rho+t+u < \pi$ and $u < \rho + t$ imply that $\sin u < \sin(\rho+t)$, hence $g(\theta) \geq g(0) = \cos d(z,y) - \cos d(x,y) > 0$. The proof is complete. □

Proof of Proposition 3.8. By Lemma 3.10, it suffices to consider the case when $\cot u < 2\cot\rho - \cot(\rho+t)$. Let $p \in \operatorname{argmin} h_{x,z}$, then by (9) and the spherical law of cosines in $\triangle apx$,

$$\cos d(x,p) = \frac{2\cos(\rho+t)\sin u + \sin^2\rho \sin(\rho+t-u)}{2\cos\rho\sin u}. \tag{15}$$

Now let $u = \rho - v$, then $\rho + t - u = t + v$. So that (8) and (15) become

$$\cos d(z,p) = \frac{2\cos(\rho-v)\sin(\rho+t) - \sin^2\rho\sin(t+v)}{2\cos\rho\sin(\rho+t)}. \tag{16}$$

$$\cos d(x,p) = \frac{2\cos(\rho+t)\sin(\rho-v)+\sin^2\rho\sin(t+v)}{2\cos\rho\sin(\rho-v)}. \tag{17}$$

It follows that

$$\begin{aligned}\cos d(z,p)\cos d(x,p) = &\,(4\cos(\rho-v)\sin(\rho+t)\cos(\rho+t)\sin(\rho-v)\\ &+2\sin^2\rho\sin^2(t+v)-\sin^4\rho\sin^2(t+v))\\ &/(4\cos^2\rho\sin(\rho+t)\sin(\rho-v)).\end{aligned} \tag{18}$$

On the other hand, (3) and (16) yield that

$$\begin{aligned}\sin d(z,p)\sin d(x,p) &= (1-\cos^2 d(z,p))\frac{\sin(\rho+t)}{\sin(\rho-v)}\\ &= (4\sin^2(\rho+t)(\cos^2\rho-\cos^2(\rho-v))-\sin^4\rho\sin^2(t+v)\\ &\quad +4\sin^2\rho\sin(t+v)\sin(\rho+t)\cos(\rho-v))\\ &\quad /(4\cos^2\rho\sin(\rho+t)\sin(\rho-v)).\end{aligned} \tag{19}$$

Then by (18) and (19) we obtain

$$\begin{aligned}&4\cos^2\rho\sin(\rho+t)\sin(\rho-v)(\cos(d(x,p)-d(z,p))-\cos(t+v))\\ =&\,4\cos^2\rho\sin(\rho+t)\sin(\rho-v)(\cos d(x,p)\cos d(z,p)+\sin d(x,p)\sin d(z,p)-\cos(t+v))\\ =&\,4\cos(\rho-v)\sin(\rho-v)\cos(\rho+t)\sin(\rho+t)+2\sin^2\rho\cos^2\rho\sin^2(t+v)\\ &+4\sin^2(\rho+t)(\cos^2\rho-\cos^2(\rho-v))+4\sin^2\rho\sin(\rho+t)\cos(\rho-v)\sin(t+v))\\ &-4\cos^2\rho\sin(\rho+t)\sin(\rho-v)\cos(t+v)\\ =&\,(-4\cos^4\rho\cos^2 v\sin^2 t-4\cos^4\rho\sin^2 t\sin^2 v+4\cos^4\rho\sin^2 t)\\ &+(-8\cos^3\rho\cos t\cos^2 v\sin\rho\sin t-8\cos^3\rho\cos t\sin\rho\sin t\sin^2 v+8\cos^3\rho\cos t\sin\rho\sin t)\\ &+(-4\cos^2\rho\cos^2 t\cos^2 v\sin^2\rho-2\cos^2\rho\cos^2 t\sin^2\rho\sin^2 v+4\cos^2\rho\cos^2 t\sin^2\rho)\\ &+4\cos^2\rho\sin^2\rho\sin v\cos v\sin t\cos t+2\cos^2\rho\cos^2 v\sin^2\rho\sin^2 t\\ =&\,2\cos^2\rho\sin^2\rho\cos^2 t\sin^2 v+4\cos^2\rho\sin^2\rho\sin v\cos v\sin t\cos t+2\cos^2\rho\cos^2 v\sin^2\rho\sin^2 t\\ =&\,2\cos^2\rho\sin^2\rho\sin^2(t+v).\end{aligned}$$

As a result,

$$\begin{aligned}\cos(d(x,p)-d(z,p)) &= \cos(t+v)+\frac{2\cos^2\rho\sin^2\rho\sin^2(t+v)}{4\cos^2\rho\sin(\rho+t)\sin(\rho-v)}\\ &= \cos(t+v)+\frac{\sin^2\rho\sin^2(t+v)}{2\sin(\rho+t)\sin(\rho-v)}\\ &= \cos(\rho+t-u)+\frac{\sin^2\rho\sin^2(\rho+t-u)}{2\sin u\sin(\rho+t)}.\end{aligned}$$

Now it suffices to use Lemma 3.11 to finish the proof. □

We also need the following lemma.

Lemma 3.12. *Let κ be real number and $1/2 < \alpha \le 1$. For $t \in (0, \rho/(2\alpha-1)]$ define*

$$F_{\alpha,\rho,\kappa}(t) = \begin{cases} \cot(\sqrt{\kappa}(2\alpha-1)t) - \cot(\sqrt{\kappa}t) - 2\cot(\sqrt{\kappa}\rho), & \text{if } \kappa > 0; \\ (1-\alpha)\rho - (2\alpha-1)t, & \text{if } \kappa = 0; \\ \coth(\sqrt{-\kappa}(2\alpha-1)t) - \coth(\sqrt{-\kappa}t) - 2\coth(\sqrt{-\kappa}\rho), & \text{if } \kappa < 0. \end{cases}$$

Assume that $1/2 < \alpha < 1$, then there exists a unique $t_\kappa \in (0, \rho/(2\alpha-1))$ such that

$$\left\{ t \in (0, \frac{\rho}{2\alpha-1}] : \ F_{\alpha,\rho,\kappa}(t) \ge 0 \right\} = (0, t_\kappa].$$

In this case, when $\kappa \le 0$, the function $F_{\alpha,\rho,\kappa}$ is strictly deceasing.

Proof. We only prove the case when $\kappa = 1$, since the proof of the other two cases are similar and easier. Observe that $F_{\alpha,\rho,1}(0+) = +\infty$ (since $1/2 < \alpha < 1$) and $F_{\alpha,\rho,1}(\rho/(2\alpha-1)) < 0$ (since $2\alpha\rho/(2\alpha-1) < \pi$), then there exists some $t_1 \in (0, \rho/(2\alpha-1))$ such that $F_{\alpha,\rho,1}(t_1) = 0$. Moreover,

$$F'_{\alpha,\rho,1}(t) = \frac{1}{\sin^2((2\alpha-1)t)} \left((\frac{\sin((2\alpha-1)t)}{\sin t})^2 - (2\alpha-1) \right).$$

Observe that the function $l(t) = \sin((2\alpha-1)t)/\sin t$ is strictly increasing on $(0, \pi/(2\alpha)]$, $l^2(0+) = (2\alpha-1)^2 < 2\alpha-1$ and $l^2(\pi/(2\alpha)) = 1 > 2\alpha-1$. Hence there exists a unique $s \in (0, \pi/(2\alpha))$ such that if $t < s$, then $F'_{\alpha,\rho,1}(t) < 0$; if $t = s$, then $F'_{\alpha,\rho,1}(t) = 0$; if $t > s$, then $F'_{\alpha,\rho,1}(t) > 0$. Hence $F_{\alpha,\rho,1}$ is strictly decreasing on $(0, s]$ and strictly increasing on $[s, \pi/(2\alpha)]$. Since $\rho/(2\alpha-1) < \pi/(2\alpha)$ and $F_{\alpha,\rho,1}(\rho/(2\alpha-1)) < 0$, the point t_1 must be unique. Moreover, it is easily seen that $\{F_{\alpha,\rho,1} \ge 0\} = (0, t_1]$. The proof is complete. □

The main theorem of this section is justified by the lemma below.

Lemma 3.13. *Assumption 3.5 implies that*

$$\frac{\alpha S_\Delta(\rho)}{\sqrt{2\alpha-1}} < S_\Delta(\frac{r_*}{2}), \ \text{where } S_\Delta(t) := \begin{cases} \sin(\sqrt{\Delta}\, t), & \text{if } \Delta > 0; \\ t, & \text{if } \Delta = 0; \\ \sinh(\sqrt{-\Delta}\, t), & \text{if } \Delta < 0. \end{cases}$$

Proof. We only prove the case when $\Delta > 0$, since the proof for the cases when $\Delta \le 0$ are easier. Without loss of generality, we can assume that $\Delta = 1$. Since $(2\alpha-1)r_*/(2\alpha) \le \pi/2$, we have

$$\frac{2\alpha\rho}{2\alpha-1} < r_* \Longleftrightarrow \sin\rho < \sin\frac{2\alpha-1}{2\alpha} r_*.$$

So that it is sufficient to show

$$\frac{\alpha}{\sqrt{2\alpha-1}} \sin\frac{2\alpha-1}{2\alpha} r_* < \sin\frac{r_*}{2}.$$

To this end, let $c = r_*/2 \in (0, \pi/2]$, we will show that the function

$$f(\alpha) = \frac{\alpha}{\sqrt{2\alpha-1}} \sin\frac{2\alpha-1}{\alpha} c$$

is strictly increasing for $\alpha \in (1/2, 1)$. Easy computation gives that

$$f'(\alpha) > 0 \Longleftrightarrow \frac{\tan(\theta c)}{\theta c} < \frac{2-\theta}{1-\theta},$$

where $\theta = (2\alpha - 1)/\alpha \in (0, 1)$. Observe that the function $x \longmapsto \tan x/x$ is increasing on $[0, \pi/2)$, hence it suffices to show that

$$\frac{\tan(\theta\pi/2)}{(\theta\pi/2)} < \frac{2-\theta}{1-\theta}.$$

This is true because Becker-Stark inequality (see [4]) yields

$$\frac{\tan(\theta\pi/2)}{(\theta\pi/2)} < \frac{1}{1-\theta^2} < \frac{2-\theta}{1-\theta}.$$

The proof is complete. $\square$

Now we are ready to give the main result of this section.

Theorem 3.14. *The following estimations hold:*

i) If $\Delta > 0$ and $Q_\mu \subset \bar{B}(a, r_*/2)$, then

$$Q_\mu \subset \bar{B}\Big(a, \frac{1}{\sqrt{\Delta}} \arcsin\big(\frac{\alpha \sin(\sqrt{\Delta}\rho)}{\sqrt{2\alpha - 1}}\big)\Big).$$

Moreover, any of the two conditions below implies $Q_\mu \subset \bar{B}(a, r_*/2)$:

$$a)\quad \frac{2\alpha\rho}{2\alpha - 1} \leq \frac{r_*}{2}; \qquad b)\quad \frac{2\alpha\rho}{2\alpha - 1} > \frac{r_*}{2} \quad \text{and} \quad F_{\alpha,\rho,\Delta}(\frac{r_*}{2} - \rho) \leq 0.$$

ii) If $\Delta = 0$, then

$$Q_\mu \subset \bar{B}\Big(a, \frac{\alpha\rho}{\sqrt{2\alpha - 1}}\Big).$$

iii) If $\Delta < 0$, then

$$Q_\mu \subset \bar{B}\Big(a, \frac{1}{\sqrt{-\Delta}} \operatorname{arcsinh}\big(\frac{\alpha \sinh(\sqrt{-\Delta}\rho)}{\sqrt{2\alpha - 1}}\big)\Big).$$

Finally, Lemma 3.13 ensures that any of the above three closed balls is contained in the open ball $B(a, r_*/2)$.

Proof. Firstly, we consider the case when $\Delta > 0$. Without loss of generality, we can assume that $\Delta = 1$. For every $x \in B_* \setminus \bar{B}(a, \rho)$, let $t_x = d(a, x) - \rho \in (0, \rho/(2\alpha - 1)]$. By Propositions 3.1 and 3.7, if there exists some z on the minimal geodesic joining x and a such that $u_z = d(a, z) \in [0, \rho + t_x)$ verifies $\rho + t_x + u_z < r_*$ and $\min_{\bar{B}(\bar{a},\rho)} \bar{h}_{\bar{x},\bar{z}} > (1 - \alpha)(\rho + t_x - u_z)/\alpha$, then $x \notin Q_\mu$. Or equivalently,

$$\begin{aligned} &Q_\mu \cap \bar{B}(a, \rho)^c \\ &\subset \Big\{ x \in B_* \setminus \bar{B}(a, \rho) : \ t_x \in (0, \frac{\rho}{2\alpha - 1}] \text{ has the property that for every } u_z \in [0, \rho + t_x) \\ &\qquad \text{such that } \rho + t_x + u_z < r_*, \min \bar{h}_{\bar{x},\bar{z}} \leq \frac{1-\alpha}{\alpha}(\rho + t_x - u_z) \Big\} := A. \end{aligned}$$

Since the restrictive condition of the set A is only on t_x, for simplicity and without ambiguity, by dropping the subscripts of t_x and u_z we rewrite A in the following form:

$$\begin{aligned}
&\Big\{t \in (0, \frac{\rho}{2\alpha-1}] : \text{ for every } u \in [0, \rho+t) \text{ such that } \rho+t+u < r_*, \\
&\qquad\qquad \min \bar{h}_{\bar{x},\bar{z}} \leq \frac{1-\alpha}{\alpha}(\rho+t-u)\Big\} \\
=& \Big\{t \in (0, \frac{r_*}{2}-\rho] : \text{ for every } u \in [0, \rho+t) \text{ such that } \rho+t+u < r_*, \\
&\qquad\qquad \min \bar{h}_{\bar{x},\bar{z}} \leq \frac{1-\alpha}{\alpha}(\rho+t-u)\Big\} \\
&\cup \Big\{t \in (\frac{r_*}{2}-\rho, \frac{\rho}{2\alpha-1}] : \text{ for every } u \in [0, \rho+t) \text{ such that } \rho+t+u < r_*, \\
&\qquad\qquad \min \bar{h}_{\bar{x},\bar{z}} \leq \frac{1-\alpha}{\alpha}(\rho+t-u)\Big\} := B \cup C.
\end{aligned}$$

Observe that for $t \in (0, r_*/2-\rho]$ and $u \in [0, \rho+t)$, we always have $\rho+t+u < r_*$, hence by Proposition 3.8 and Lemma 3.11,

$$\begin{aligned}
B =& \Big\{t \in (0, \frac{r_*}{2}-\rho] : \text{ for every } u \in [0, \rho+t), \ \min \bar{h}_{\bar{x},\bar{z}} \leq \frac{1-\alpha}{\alpha}(\rho+t-u)\Big\} \\
=& \Big\{t \in (0, \frac{r_*}{2}-\rho] : \text{ for every } u \in [0, \rho+t) \text{ such that } \cot u \geq 2\cot\rho - \cot(\rho+t), \\
&\qquad\qquad t-\rho+u \leq \frac{1-\alpha}{\alpha}(\rho+t-u)\Big\} \\
&\cap \Big\{t \in (0, \frac{r_*}{2}-\rho] : \text{ for every } u \in [0, \rho+t) \text{ such that } \cot u < 2\cot\rho - \cot(\rho+t), \\
&\qquad\qquad \cos(\rho+t-u) + \frac{\sin^2\rho \sin^2(\rho+t-u)}{2\sin u \sin(\rho+t)} \geq \cos\big(\frac{1-\alpha}{\alpha}(\rho+t-u)\big)\Big\} \\
=& \Big\{t \in (0, \frac{r_*}{2}-\rho] : \text{ for every } u \in [0, \rho+t) \text{ such that } \cot u \geq 2\cot\rho - \cot(\rho+t), \\
&\qquad\qquad u \leq \rho - (2\alpha-1)t\Big\} \\
&\cap \Big\{t \in (0, \frac{r_*}{2}-\rho] : \text{ for every } u \in [0, \rho+t) \text{ such that } \cot u < 2\cot\rho - \cot(\rho+t), \\
&\qquad\qquad \sin(\rho+t) \leq \frac{\sin^2\rho}{4\sin u} \frac{\sin(\rho+t-u)}{\sin\frac{\rho+t-u}{2\alpha}} \frac{\sin(\rho+t-u)}{\sin\big(\frac{2\alpha-1}{2\alpha}(\rho+t-u)\big)}\Big\} \\
=& \Big\{t \in (0, \frac{r_*}{2}-\rho] : \ \cot(\rho-(2\alpha-1)t) \leq 2\cot\rho - \cot(\rho+t)\Big\}
\end{aligned}$$

$$\cap \left\{ t \in (0, \frac{r_*}{2} - \rho] : \text{ for every } u \in [0, \rho + t) \text{ such that } \cot u < 2\cot\rho - \cot(\rho + t), \right.$$

$$\left. \sin(\rho + t) \leq \frac{\sin^2\rho}{4\sin u} \frac{\sin(\rho + t - u)}{\sin \dfrac{\rho + t - u}{2\alpha}} \frac{\sin(\rho + t - u)}{\sin\big(\dfrac{2\alpha - 1}{2\alpha}(\rho + t - u)\big)} \right\}$$

$$\subset \left\{ t \in (0, \frac{r_*}{2} - \rho] : \text{ for every } u \in (\rho - (2\alpha - 1)t, \rho + t), \right.$$

$$\left. \sin(\rho + t) \leq \frac{\sin^2\rho}{4\sin u} \frac{\sin(\rho + t - u)}{\sin \dfrac{\rho + t - u}{2\alpha}} \frac{\sin(\rho + t - u)}{\sin\big(\dfrac{2\alpha - 1}{2\alpha}(\rho + t - u)\big)} \right\}$$

$$\subset \left\{ t \in (0, \frac{r_*}{2} - \rho] : \sin(\rho + t) \leq \frac{\sin^2\rho}{4\sin(\rho + t)} \cdot 2\alpha \cdot \frac{2\alpha}{2\alpha - 1} \right\}$$

$$= \left\{ t \in (0, \frac{r_*}{2} - \rho] : \rho + t \leq \arcsin\big(\frac{\alpha \sin\rho}{\sqrt{2\alpha - 1}}\big) \right\}. \tag{20}$$

Hence if $Q_\mu \subset \bar{B}(a, r_*/2)$, then $C = \phi$ and (20) says that

$$Q_\mu \subset \bar{B}\left(a, \arcsin\big(\frac{\alpha \sin\rho}{\sqrt{2\alpha - 1}}\big)\right),$$

this completes the proof of the first assertion of i). To show the second one, observe that if a) holds, then Theorem 3.2 implies the desired result. Hence assume that b) holds. By Proposition 3.8 one has

$$C \subset \left\{ t \in (\frac{r_*}{2} - \rho, \frac{\rho}{2\alpha - 1}] : \text{ for every } u \in [0, r_* - (\rho + t)) \text{ such that} \right.$$

$$\cot u \geq 2\cot\rho - \cot(\rho + t),$$

$$\left. t - \rho + u \leq \frac{1 - \alpha}{\alpha}(\rho + t - u) \right\}$$

$$= \left\{ t \in (\frac{r_*}{2} - \rho, \frac{\rho}{2\alpha - 1}] : \cot(\rho - (2\alpha - 1)t) \leq 2\cot\rho - \cot(\rho + t) \right\}.$$

Observe that for $t \in (r_*/2 - \rho,\ \rho/(2\alpha - 1))$ we have

$$\begin{aligned}
&\cot(\rho - (2\alpha - 1)t) \leq 2\cot\rho - \cot(\rho + t) \\
\Longleftrightarrow\ &\cot(\rho - (2\alpha - 1)t) - \cot\rho \leq \cot\rho - \cot(\rho + t) \\
\Longleftrightarrow\ &\frac{\sin(\rho - (2\alpha - 1)t)}{\sin((2\alpha - 1)t)} \geq \frac{\sin(\rho + t)}{\sin t} \\
\Longleftrightarrow\ &\cot((2\alpha - 1)t) - \cot\rho \geq \cot\rho + \cot t \\
\Longleftrightarrow\ &F_{\alpha,\rho,1}(t) \geq 0.
\end{aligned}$$

Hence

$$C \subset \left\{ t \in (\frac{r_*}{2} - \rho, \frac{\rho}{2\alpha - 1}] : F_{\alpha,\rho,1}(t) \geq 0 \right\} := D.$$

If $\alpha = 1$, clearly $D = \phi$. Now let $1/2 < \alpha < 1$, then Lemma 3.12 yields that

$$D = (\frac{r_*}{2} - \rho, \frac{\rho}{2\alpha - 1}] \cap (0, t_1] = \phi.$$

Thus $C = \phi$ still holds, that is, $Q_\mu \subset \bar{B}(a, r_*/2)$. The proof of $i)$ is complete.

Now let us turn to the proof of $ii)$ and $iii)$. In fact, the proof for these two cases are essentially the same as that of $i)$ except to note that we no longer need to assume that $Q_\mu \subset \bar{B}(a, r_*/2)$, because this is implied by Assumption 3.5. To see this, if $4\alpha\rho \le (2\alpha - 1)r_*$, then it suffices to use Theorem 3.2. So that let us assume $4\alpha\rho > (2\alpha - 1)r_*$ and show that $F_{\alpha,\rho,\Delta}(r_*/2 - \rho) \le 0$ for $\Delta \in \{-1, 0\}$. This is trivial if $\Delta = 0$ or $\alpha = 1$, hence let $\Delta = -1$ and $\alpha \in (1/2, 1)$. Since $r_*/2 - \rho > (1 - \alpha)\rho/(2\alpha - 1)$ and $F_{\alpha,\rho,-1}$ is strictly decreasing, it suffices to show that $F_{\alpha,\rho,-1}((1 - \alpha)\rho/(2\alpha - 1)) \le 0$. To this end, define $f(\alpha) = F_{\alpha,\rho,-1}((1 - \alpha)\rho/(2\alpha - 1))$, easy computation gives that

$$f'(\alpha) = \frac{\rho}{\sinh^2((1-\alpha)\rho)} - \frac{\rho}{(2\alpha - 1)^2 \sinh^2(\frac{1-\alpha}{2\alpha-1}\rho)} > 0,$$

because the function $x \longmapsto \sinh x/x$ is strictly increasing. Hence $f(\alpha) < f(1-) = 2(\rho^{-1} - \coth\rho) < 0$. The proof is complete. □

Remark 3.15. When $\Delta > 0$, Assumption 3.5 does not imply the condition $b)$ in $i)$. In fact, in the case when $M = \mathbb{S}^2$, we have $r_* = \pi$ and $\Delta = 1$. Then let $\alpha = 0.51$ and $\rho = 0.99\pi(1 - (2\alpha)^{-1})$, then $2\alpha\rho/(2\alpha - 1) \in (\pi/2 + 1.5393, \pi - 0.0314)$, but $F_{\alpha,\rho,1}(\pi/2 - \rho) \approx 0.2907 > 0$.

Remark 3.16. It is easily seen that if we replace r_* by any $r \in (0, r_*]$ in Assumption 3.5, then Lemma 3.13 still holds when r_* is replaced by r. This observation can be used to reinforce the conclusions of Theorem 3.14. For example, in the case when $\Delta > 0$,

$$\frac{2\alpha\rho}{2\alpha - 1} \le \frac{r_*}{2} \text{ implies that } Q_\mu \subset \bar{B}\left(a, \frac{1}{\sqrt{\Delta}} \arcsin\big(\frac{\alpha\sin(\sqrt{\Delta}\rho)}{\sqrt{2\alpha - 1}}\big)\right) \subset B\left(a, \frac{r_*}{4}\right).$$

Remark 3.17. Although we have chosen the framework of this section to be a Riemannian manifold, the essential tool that has been used is the hinge version of the triangle comparison theorem. Consequently, all the results in this section remain true if M is a CAT(Δ) space (see [6, Chapter 2]) and r_* is replaced by $\pi/\sqrt{\Delta}$ in Assumption 3.5.

Remark 3.18. For the case when $\alpha = 1$, Assumption 3.5 becomes

$$\rho < \frac{1}{2}\min\{\frac{\pi}{\sqrt{\Delta}}, \text{inj}\}.$$

Observe that in this case, when $\Delta > 0$, the condition $F_{1,\rho,\Delta}(r_*/2 - \rho) \le 0$ is trivially true in case of need. Hence Theorem 3.14 yields that $Q_\mu \subset \bar{B}(a, \rho)$, which is exactly what the Theorem 2.1 in [1] says for medians.

4. UNIQUENESS OF FRÉCHET SAMPLE MEDIANS IN COMPACT RIEMANNIAN MANIFOLDS

In this section, we shall always assume that M is a complete Riemannian manifold of dimension $l \ge 2$. The Riemannian metric and the Riemannian distance are denoted by $\langle \cdot, \cdot \rangle$ and d, respectively. For each point $x \in M$, S_x denotes the unit sphere in T_xM. Moreover, for a tangent vector $v \in S_x$, the distance between x and its cut point along the geodesic starting from x with velocity v is denoted by $\tau(v)$. Certainly, if there is no cut point along this geodesic, then we define $\tau(v) = +\infty$.

For every point $(x_1, \dots, x_N) \in M^N$, where $N \geq 3$ is a fixed natural number, we write

$$\mu(x_1, \dots, x_N) = \frac{1}{N} \sum_{k=1}^{N} \delta_{x_k}.$$

The set of all the Fréchet medians of $\mu(x_1, \dots, x_N)$ is denoted by $Q(x_1, \dots, x_N)$.

We begin with the basic observation that if one data point is moved towards a median along some minimizing geodesic for a little distance, then the median remains unchanged.

Proposition 4.1. *Let $(x_1, \dots, x_N) \in M^N$ and $m \in Q(x_1, \dots, x_N)$. Fix a normal geodesic $\gamma : [0, +\infty) \to M$ such that $\gamma(0) = x_1$, $\gamma(d(x_1, m)) = m$. Then for every $t \in [0, d(x_1, m)]$ we have*

$$Q(\gamma(t), x_2, \dots, x_N) = \begin{cases} Q(x_1, \dots, x_N) \cap \gamma[t, \tau(\dot\gamma(0))], & \text{if } \tau(\dot\gamma(0)) < +\infty; \\ Q(x_1, \dots, x_N) \cap \gamma_1[t, +\infty), & \text{if } \tau(\dot\gamma(0)) = +\infty. \end{cases}$$

Particularly, $m \in Q(\gamma(t), x_2, \dots, x_N)$.

Proof. For simplicity, let $\mu = \mu(x_1, \dots, x_N)$ and $\mu_t = \mu(\gamma(t), x_2, \dots, x_N)$. Then for every $x \in M$,

$$\begin{aligned} f_{\mu_t}(x) - f_{\mu_t}(m) &= \Big(f_\mu(x) - \frac{1}{N} d(x, x_1) + \frac{1}{N} d(x, \gamma(t)) \Big) \\ &\quad - \Big(f_\mu(m) - \frac{1}{N} d(m, x_1) + \frac{1}{N} d(m, \gamma(t)) \Big) \\ &= \Big(f_\mu(x) - f_\mu(m) \Big) + \Big(d(x, \gamma(t)) + t - d(x, x_1) \Big) \geq 0. \end{aligned}$$

So that $m \in Q_{\mu_t}$. Combine this with the fact that m is a median of μ, it is easily seen from the above proof that

$$Q_{\mu_t} = Q_\mu \cap \{x \in M : d(x, \gamma(t)) + t = d(x, x_1)\}.$$

Now the conclusion follows from the definition of $\tau(\dot\gamma(0))$. □

The following theorem states that in order to get the uniqueness of Fréchet medians, it suffices to move two data points towards a common median along some minimizing geodesics for a little distance.

Theorem 4.2. *Let $(x_1, \dots, x_N) \in M^N$ and $m \in Q(x_1, \dots, x_N)$. Fix two normal geodesics $\gamma_1, \gamma_2 : [0, +\infty) \to M$ such that $\gamma_1(0) = x_1$, $\gamma_1(d(x_1, m)) = m$, $\gamma_2(0) = x_2$ and $\gamma_2(d(x_2, m)) = m$. Assume that*

$$x_2 \notin \begin{cases} \gamma_1[0, \tau(\dot\gamma_1(0))], & \text{if } \tau(\dot\gamma_1(0)) < +\infty; \\ \gamma_1[0, +\infty), & \text{if } \tau(\dot\gamma_1(0)) = +\infty. \end{cases}$$

Then for every $t \in (0, d(x_1, m)]$ and $s \in (0, d(x_2, m)]$ we have

$$Q(\gamma_1(t), \gamma_2(s), x_3, \dots, x_N) = \{m\}.$$

Proof. Without loss of generality, we may assume that both $\tau(\dot\gamma_1(0))$ and $\tau(\dot\gamma_2(0))$ are finite. Applying Proposition 4.1 two times we get

$$Q(\gamma_1(t), \gamma_2(s), x_3, \dots, x_N) \subset Q(x_1, \dots, x_N) \cap \gamma_1[t, \tau(\dot\gamma_1(0))] \cap \gamma_2[s, \tau(\dot\gamma_2(0))].$$

Since $x_2 \notin \gamma_1[0, \tau(\dot\gamma_1(0))]$, the definition of cut point yields $\gamma_1[t, \tau(\dot\gamma_1(0))] \cap \gamma_2[s, \tau(\dot\gamma_2(0))] = \{m\}$. The proof is complete. □

We need the following necessary conditions of Fréchet medians.

Proposition 4.3. *Let $(x_1, \dots, x_N) \in M^N$ and $m \in Q(x_1, \dots, x_N)$. For every $k = 1, \dots, N$ let $\gamma_k : [0, d(m, x_k)] \to M$ be a normal geodesic such that $\gamma_k(0) = m$ and $\gamma_k(d(m, x_k)) = x_k$.*
i) If m does not coincide with any x_k, then

$$\sum_{k=1}^{N} \dot\gamma_k(0) = 0. \tag{21}$$

In this case, the minimizing geodesics $\gamma_1, \dots, \gamma_N$ are uniquely determined.
ii) If m coincides with some x_{k_0}, then

$$\left| \sum_{x_k \neq x_{k_0}} \dot\gamma_k(0) \right| \leq \sum_{x_k = x_{k_0}} 1. \tag{22}$$

Proof. For sufficiently small $\varepsilon > 0$, Proposition 4.1 yields that m is a median of $\mu(\gamma_1(\varepsilon), \dots, \gamma_N(\varepsilon))$. Hence [19, Theorem 2.2] gives (21) and (22). Now assume that m does not coincide with any x_k and, without loss of generality, there is another normal geodesic $\zeta_1 : [0, d(m, x_1)] \to M$ such that $\zeta_1(0) = m$ and $\zeta_1(d(m, x_1)) = x_1$. Then (21) yields $\dot\zeta_1(0) + \sum_{k=2}^{N} \dot\gamma_k(0) = 0$. So that $\dot\zeta_1(0) = \dot\gamma_1(0)$, that is to say, $\zeta_1 = \gamma_1$. The proof is complete. □

From now on, we will only consider the case when M is a compact Riemannian manifold. As a result, let the following assumption hold in the rest part of this section:

Assumption 4.4. M is a compact Riemannian manifold with diameter L.

In what follows, all the measure-theoretic statements should be understood to be with respect to the canonical Lebesgue measure of the underlying manifold. Let λ_M denote the canonical Lebesgue measure of M.

The following lemma gives a simple observation on dimension.

Lemma 4.5. *The manifold*

$$V = \Big\{ (x, n_1, \dots, n_N) : x \in M,\ (n_k)_{1 \leq k \leq N} \subset S_x,\ \sum_{k=1}^{N} n_k = 0 \text{ and there exist } k_1 \neq k_2 \text{ such that } n_{k_1} \text{ and } n_{k_2} \text{ are linearly independent} \Big\}$$

is of dimension $N(l-1)$.

Proof. Consider the manifold

$$W = \Big\{ (x, n_1, \dots, n_N) : x \in M,\ (n_k)_{1 \leq k \leq N} \subset S_x \text{ and there exist } k_1 \neq k_2 \text{ such that } n_{k_1} \text{ and } n_{k_2} \text{ are linearly independent} \Big\},$$

then the smooth map

$$f: \quad W \longrightarrow TM, \quad (x, n_1, \dots, n_N) \longmapsto \sum_{k=1}^{N} n_k$$

has full rank l everywhere on W. Now the constant rank level set theorem (see [14, Theorem 8.8]) yields the desired result. $\square$

Our method of studying the uniqueness of Fréchet medians is based on the regularity properties of the following function:

$$\begin{aligned}\varphi: \quad & V \times \mathbf{R}^N \longrightarrow M^N, \\ & (x, n_1, \dots, n_N, r_1, \dots, r_N) \longmapsto (\exp_x(r_1 n_1), \dots, \exp_x(r_N n_N)).\end{aligned}$$

For a closed interval $[a, b] \subset \mathbf{R}$, the restriction of φ to $V \times [a, b]^N$ will be denoted by $\varphi_{a,b}$. The canonical projection of $V \times \mathbf{R}^N$ onto M and $\mathbf{R}^N$ will be denoted by σ and ζ, respectively.

Generally speaking, the non uniqueness of Fréchet medians is due to some symmetric properties of data points. As a result, generic data points should have a unique Fréchet median. In mathematical language, this means that the set of all the particular positions of data points is of measure zero. Now our aim is to find all these particular cases. Firstly, in view of the uniqueness result of Riemannian medians (see [19, Theorem 3.1]), the first null set that should be eliminated is

$$C_1 = \Big\{ (x_1, \dots, x_N) \in M^N : x_1, \dots, x_N \text{ are contained in a single geodesic} \Big\}.$$

Observe that C_1 is a closed subset of M^N. The second null set coming into our sight is the following one:

$$C_2 = \Big\{ (x_1, \dots, x_N) \in M^N : \ (x_1, \dots, x_N) \text{ is a critical value of } \varphi_{0,L} \Big\}.$$

Since φ is smooth, Sard's theorem implies that C_2 is of measure zero. Moreover, it is easily seen that $(M^N \setminus C_1) \cap C_2$ is closed in $M^N \setminus C_1$.

The following proposition says that apart form $C_1 \cup C_2$ one can only have a finite number of Fréchet medians.

Proposition 4.6. *$Q(x_1, \dots, x_N)$ is a finite set for every $(x_1, \dots, x_N) \in M^N \setminus (C_1 \cup C_2)$.*

Proof. Let $(x_1, \dots, x_N) \in M^N \setminus (C_1 \cup C_2)$ and $A \subset Q(x_1, \dots, x_N)$ be the set of medians that do not coincide with any x_k. If $A = \phi$, then there is nothing to prove. Now assume that $A \neq \phi$, then Proposition 4.3 implies that $A \subset \sigma \circ \varphi_{0,L}^{-1}(x_1, \dots, x_N)$. Moreover, Lemma 4.5 and the constant rank level set theorem imply that $\varphi_{0,L}^{-1}(x_1, \dots, x_N)$ is a zero dimensional regular submanifold of $V \times [0, L]^N$, that is, some isolated points. Since $(x_1, \dots, x_N) \notin C_1$, $\varphi_{0,L}^{-1}(x_1, \dots, x_N)$ is also compact, hence it is a finite set. So that A is also finite, as desired. $\square$

The following two lemmas enable us to avoid the problem of cut locus.

Lemma 4.7. *Let U be a bounded open subset of $V \times \mathbf{R}^N$ such that $\varphi : U \longmapsto \varphi(U)$ is a diffeomorphism, then*

$$\lambda_M^{\otimes N}\left\{(x_1,\dots,x_N) \in \varphi(U) : \sigma\circ\varphi^{-1}(x_1,\dots,x_N) \in \bigcup_{k=1}^N \mathrm{Cut}(x_k)\right\} = 0.$$

Proof. Without loss of generality, we will show that $\lambda_M^{\otimes N}\{(x_1,\dots,x_N) \in M^N : \sigma\circ\varphi^{-1}(x_1,\dots,x_N) \in \mathrm{Cut}(x_N)\} = 0$. In fact, letting $(x_1,\dots,x_N) = \varphi(x, n_1, \dots n_N, r_1, \dots, r_N)$, $\mathbf{n} = (n_1,\dots n_N)$ and $\det(D\varphi) \le c$ on U for some $c > 0$, then the change of variable formula and Fubini's theorem yield that

$$\begin{aligned}
&\lambda_M^{\otimes N}\{(x_1,\dots,x_N) \in \varphi(U) : \sigma\circ\varphi^{-1}(x_1,\dots,x_N) \in \mathrm{Cut}(x_N)\}\\
=&\lambda_M^{\otimes N}\{(x_1,\dots,x_N) \in \varphi(U) : x_N \in \mathrm{Cut}(\sigma\circ\varphi^{-1}(x_1,\dots,x_N))\}\\
=&\int_{\varphi(U)} \mathbf{1}_{\{x_N\in\mathrm{Cut}(\sigma\circ\varphi^{-1}(x_1,\dots,x_N))\}} dx_1\dots dx_N\\
=&\int_U \mathbf{1}_{\{\exp_x(r_N n_N)\in\mathrm{Cut}(x)\}} \det(D\varphi) dx\, d\mathbf{n}\, dr_1\dots dr_N\\
\le& c\int_{V\times\mathbf{R}^N} \mathbf{1}_{\{\exp_x(r_N n_N)\in\mathrm{Cut}(x)\}} dx\, d\mathbf{n}\, dr_1\dots dr_N\\
=& c\int_{V\times\mathbf{R}^{N-1}} dx\, d\mathbf{n}\, dr_1\dots dr_{N-1}\int_{\mathbf{R}} \mathbf{1}_{\{\exp_x(r_N n_N)\in\mathrm{Cut}(x)\}} dr_N\\
=&0.
\end{aligned}$$

The proof is complete. □

In order to tackle the cut locus, it is easily seen that the following null set should be eliminated:

$$C_3 = \left\{(x_1,\dots,x_N) \in M^N :\ x_i \in \{x_j\}\cup\mathrm{Cut}(x_j),\ \text{for some } i \ne j\right\}.$$

Observe that C_3 is also closed because the set $\{(x,y) \in M^2 :\ x \in \mathrm{Cut}(y)\}$ is closed.

Lemma 4.8. *For every $(x_1,\dots,x_N) \in M^N \setminus (C_1\cup C_2\cup C_3)$, there exists $\delta > 0$ such that*

$$\lambda_M^{\otimes N}\left\{(y_1,\dots,y_N) \in B(x_1,\delta)\times\cdots\times B(x_N,\delta) :\ Q(y_1,\dots,y_N)\cap\bigcup_{k=1}^N \mathrm{Cut}(y_k) \ne \phi\right\} = 0.$$

Proof. If $Q(x_1,\dots,x_N) \subset \{x_1,\dots,x_N\}$, then the assertion is trivial by Theorem 2.3. Now assume that $Q(x_1,\dots,x_N)\setminus\{x_1,\dots,x_N\} \ne \phi$, then the proof of Proposition 4.6 yields that $\varphi_{0,L}^{-1}(x_1,\dots,x_N)$ is finite. Hence we can choose $\varepsilon, \eta > 0$ and O a relatively compact open subset of V such that $\varepsilon < \min\{\operatorname{inj} M/2, \min\{r_k : (r_1,\dots,r_N) \in \zeta\circ\varphi_{0,L}^{-1}(x_1,\dots,x_N), k = 1,\dots,N\}\}$, $B(x_i, 2\varepsilon)\cap\mathrm{Cut}(B(x_j,2\varepsilon)) = \phi$ and $\varphi_{\varepsilon,L+\eta}^{-1}(x_1,\dots,x_N) \subset O\times(\varepsilon, L+\eta)^N$. Then by the stack of records theorem (see [11, Exercise 7, Chapter 1, Section 4]), there exists $\delta \in (0,\varepsilon)$ such that $U = B(x_1,\delta)\times\cdots\times B(x_N,\delta)$ verifies $\varphi_{\varepsilon,L+\eta}^{-1}(U) = V_1\cup\cdots\cup V_h$, $V_i\cap V_j = \phi$ for $i \ne j$ and $\varphi_{\varepsilon,L+\eta} : V_i \to U$ is a diffeomorphism for every i. Lemma 4.7 yields that there exists a null set $A \subset U$, such that for every $(y_1,\dots,y_N) \in U\setminus A$ and for

every $(y, n_1, \ldots, n_N, r_1, \ldots, r_N) \in \varphi^{-1}_{\varepsilon,L+\eta}(y_1, \ldots, y_N)$ one always has $y \notin \bigcup_{k=1}^N \mathrm{Cut}(y_k)$. Particularly, for $m \in Q(y_1, \ldots, y_N)$ such that $d(m, y_k) \geq \varepsilon$ for every k, we have $m \notin \bigcup_{k=1}^N \mathrm{Cut}(y_k)$. Now let $m \in Q(y_1, \ldots, y_N)$ such that $d(m, y_{k_0}) < \varepsilon$ for some k_0, then $d(m, x_{k_0}) \leq d(m, y_{k_0}) + d(y_{k_0}, x_{k_0}) < 2\varepsilon$. So that $m \notin \bigcup_{k=1}^N \mathrm{Cut}(y_k)$ since $y_k \in B(x_k, 2\varepsilon)$. This completes the proof. □

Now the cut locus can be eliminated without difficulty.

Proposition 4.9. *The set*

$$C_4 = \left\{(x_1, \ldots, x_N) \in M^N : \ Q(x_1, \ldots, x_N) \cap \bigcup_{k=1}^N \mathrm{Cut}(x_k) \neq \phi\right\}$$

is of measure zero and is closed.

Proof. It suffices to show that C_4 is of measure zero. This is a direct consequence of Lemma 4.8 and the fact that $M \setminus (C_1 \cup C_2 \cup C_3)$ is second countable. □

Let $x, y \in M$ such that $y \notin \{x\} \cup \mathrm{Cut}(x)$, we denote $\gamma_{xy} : [0, d(x, y)] \to M$ the unique minimizing geodesic such that $\gamma_{xy}(0) = x$ and $\gamma_{xy}(d(x, y)) = y$. For every $u \in T_xM$ and $v \in T_yM$, let $J(v, u)(\cdot)$ be the unique Jacobi field along γ_{xy} with boundary condition $J(u, v)(0) = u$ and $J(u, v)(d(x, y)) = v$.

Lemma 4.10. *Let $x, y \in M$ such that $y \notin \{x\} \cup \mathrm{Cut}(x)$. Then for every $v \in T_yM$, we have*

$$\nabla_v \frac{\exp_x^{-1}(\cdot)}{d(x, \cdot)} = \dot{J}(0_x, v^{\mathrm{nor}})(0),$$

where v^{nor} is the normal component of v with respect to $\dot{\gamma}_{xy}(d(x, y))$.

Proof. By [2, p. 1517],

$$\begin{aligned}
\nabla_v \frac{\exp_x^{-1}(\cdot)}{d(x, \cdot)} &= \frac{\nabla_v \exp_x^{-1}(\cdot)}{d(x, y)} - \frac{\exp_x^{-1} y \nabla_v d(x, \cdot)}{d(x, y)^2} \\
&= \frac{\nabla_v \exp_x^{-1}(\cdot)}{d(x, y)} - \left\langle v, \frac{-\exp_y^{-1} x}{d(x, y)}\right\rangle \frac{\exp_x^{-1} y}{d(x, y)^2} \\
&= \dot{J}(0_x, v)(0) - \dot{J}(0_x, v^{\mathrm{tan}})(0) \\
&= \dot{J}(0_x, v^{\mathrm{nor}})(0),
\end{aligned}$$

where v^{tan} is the tangent component of u with respect to $\dot{\gamma}_{xy}(d(x, y))$. □

With this differential formula, another particular case can be eliminated now.

Proposition 4.11. *The set*

$$C_5 = \left\{(x_1, \ldots x_N) \in M^N \setminus (C_1 \cup C_3) : \left| \sum_{k \neq k_0} \frac{\exp_{x_{k_0}}^{-1} x_k}{d(x_{k_0}, x_k)} \right| = 1 \textit{ for some } k_0\right\}$$

is of measure zero and is closed in $M^N \setminus (C_1 \cup C_3)$.

Proof. Without loss of generality, let us show that

$$C_5' = \Big\{(x_1, \ldots, x_N) \in M^N \setminus (C_1 \cup C_3) : h(x_1, \ldots, x_N) = 1\Big\}$$

is of measure zero, where

$$h(x_1, \ldots, x_N) = \Big| \sum_{k=1}^{N-1} \frac{\exp_{x_N}^{-1} x_k}{d(x_N, x_k)} \Big|^2.$$

By the constant rank level set theorem, it suffices to show that $\operatorname{grad} h$ is nowhere vanishing on $M^N \setminus (C_1 \cup C_3)$. To this end, let $(x_1, \ldots, x_N) \in M^N \setminus (C_1 \cup C_3)$ and $u = \sum_{k=1}^{N-1} \exp_{x_N}^{-1} x_k / d(x_N, x_k)$. Since $N \geq 3$, without loss of generality, we can assume that u and $\exp_{x_N}^{-1} x_1$ are not parallel. Then for each $v \in T_{x_1}M$, by lemma 4.10 we have

$$\nabla_v h(\cdot, x_2, \ldots, x_N) = \Big\langle \nabla_v \frac{\exp_{x_N}^{-1} x_1}{d(x_N, x_1)}, u \Big\rangle = 2\langle \dot{J}(0_{x_N}, v^{\mathrm{nor}})(0), u\rangle = 2\langle \psi(v), u\rangle,$$

where the linear map ψ is defined by

$$\psi : \quad T_{x_1}M \longrightarrow T_{x_N}M, \quad v \longmapsto \dot{J}(0_{x_N}, v^{\mathrm{nor}})(0),$$

v^{nor} is the normal component of v with respect to $\exp_{x_1}^{-1} x_N$. Hence we have $\operatorname{grad}_{x_1} h(\cdot, x_2, \ldots, x_N) = \psi^*(u)$, where ψ^* is the adjoint of ψ. Since the range space of ψ is the orthogonal complement of $\exp_{x_N}^{-1} x_1$, one has necessarily $\psi^*(u) \neq 0$, this completes the proof. □

The reason why the set C_5 should be eliminated is given by the following simple lemma.

Lemma 4.12. *Let $(x_1, \ldots, x_N), (x_1^i, \ldots, x_N^i) \in M^N \setminus C_3$ for every $i \in \mathbf{N}$ and $(x_1^i, \ldots, x_N^i) \longrightarrow (x_1, \ldots, x_N)$, when $i \longrightarrow \infty$. Assume that $m_i \in Q(x_1^i, \ldots, x_N^i) \setminus \{x_1^i, \ldots, x_N^i\}$ and $m_i \longrightarrow x_{k_0}$, then*

$$\Big| \sum_{k \neq k_0} \frac{\exp_{x_{k_0}}^{-1} x_k}{d(x_{k_0}, x_k)} \Big| = 1.$$

Proof. It suffices to note that for i sufficiently large, Proposition 4.3 gives

$$\Big| \sum_{k \neq k_0} \frac{\exp_{m_i}^{-1} x_k^i}{d(m_i, x_k^i)} \Big| = \Big| \frac{\exp_{m_i}^{-1} x_{k_0}^i}{d(m_i, x_{k_0}^i)} \Big| = 1.$$

Then letting $i \to \infty$ gives the result. □

As a corollary to Proposition 4.11, the following proposition tells us that for generic data points, there cannot exist two data points which are both Fréchet medians.

Proposition 4.13. *The set*

$$C_6 = \Big\{(x_1, \ldots, x_N) \in M^N \setminus (C_1 \cup C_3 \cup C_5) :$$

$$\textit{there exist } i \neq j \textit{ such that } f_\mu(x_i) = f_\mu(x_j), \textit{ where } \mu = \mu(x_1 \ldots, x_N)\Big\}$$

is of measure zero and is closed in $M^N \setminus (C_1 \cup C_3 \cup C_5)$.

Proof. For every $(x_1,\dots,x_N)\in M^N\setminus(C_1\cup C_3\cup C_5)$, let $f(x_1,\dots,x_N)=f_\mu(x_{N-1})$ and $g(x_1,\dots,x_N)=f_\mu(x_N)$, where $\mu=\mu(x_1\dots,x_N)$. Without loss of generality, we will show that $\{(x_1,\dots,x_N)\in M^N\setminus(C_1\cup C_3\cup C_5):f(x_1,\dots,x_N)=g(x_1,\dots,x_N)\}$ is of measure zero. Always by the constant rank level set theorem, it suffices to show that $\operatorname{grad} f$ and $\operatorname{grad} g$ are nowhere identical on $M^N\setminus(C_1\cup C_3\cup C_5)$. In fact,

$$\begin{aligned}\operatorname{grad}_{x_N} f(x_1,\dots,x_N)&=\frac{-\exp_{x_N}^{-1}x_{N-1}}{d(x_N,x_{N-1})}\\&\neq\sum_{k=1}^{N-1}\frac{-\exp_{x_N}^{-1}x_k}{d(x_N,x_k)}=\operatorname{grad}_{x_N} g(x_1,\dots,x_N),\end{aligned}$$

because C_5 is eliminated, as desired. □

As needed in the following proofs, the restriction of φ on the set

$$E=\Big\{(x,n_1,\dots,n_N,r_1,\dots,r_N)\in V\times\mathbf{R}^N:0<r_k<\tau(n_k)\text{ for every }k\Big\}$$

is denoted by let $\hat\varphi$. Clearly, $\hat\varphi$ is smooth.

The lemma below is a final preparation for the main result of this section.

Lemma 4.14. *Let U be an open subset of $M^N\setminus\bigcup_{k=1}^6 C_k$. Assume that $U_1\cup U_2\subset\hat\varphi^{-1}(U)$ such that for $i=1,2$, $\hat\varphi_i=\hat\varphi|_{U_i}:U_i\to U$ is a diffeomorphism and $\sigma(U_1)\cap\sigma(U_2)=\phi$. For simplicity, when $(x_1,\dots,x_N)\in U$, we write $x=\sigma\circ\hat\varphi_1^{-1}(x_1,\dots,x_N)$, $y=\sigma\circ\hat\varphi_2^{-1}(x_1,\dots,x_N)$ and $\mu=\mu(x_1,\dots,x_N)$. Then the following two sets are of measure zero:*

$$\Big\{(x_1,\dots,x_N)\in U:f_\mu(x)=f_\mu(y)\Big\}\ \textit{and}$$
$$\Big\{(x_1,\dots,x_N)\in U:\ \textit{there exists } k_0 \textit{ such that } f_\mu(x)=f_\mu(x_{k_0})\Big\}.$$

Proof. We only show the first set is null, since the proof for the second one is similar. Let $f_1(x_1,\dots,x_N)=f_\mu(x)$, $f_2(x_1,\dots,x_N)=f_\mu(y)$ and $w_k\in T_{x_k}M$. Then the first variational formula of arc length (see [8, p. 5]) yields that

$$\begin{aligned}&\frac{d}{dt}\Big|_{t=0} f_1(\exp_{x(t)}(tw_1),\dots\exp_{x(t)}(tw_N))\\&=\sum_{k=1}^N\left(\left\langle\frac{-\exp_{x_k}^{-1}x}{d(x_k,x)},w_k\right\rangle-\left\langle\dot x(0),\frac{\exp_x^{-1}x_k}{d(x,x_k)}\right\rangle\right)\\&=\sum_{k=1}^N\left\langle\frac{-\exp_{x_k}^{-1}x}{d(x_k,x)},w_k\right\rangle-\left\langle\dot x(0),\sum_{k=1}^N\frac{\exp_x^{-1}x_k}{d(x,x_k)}\right\rangle\\&=\sum_{k=1}^N\left\langle\frac{-\exp_{x_k}^{-1}x}{d(x_k,x)},w_k\right\rangle.\end{aligned}$$

Hence

$$\operatorname{grad} f_1(x_1,\dots,x_N)=\left(\frac{-\exp_{x_1}^{-1}x}{d(x_1,x)},\dots,\frac{-\exp_{x_N}^{-1}x}{d(x_N,x)}\right).$$

Observe that $(x_1, \ldots, x_N) \notin C_1$, $N \geq 3$ and $x \neq y$, we have grad $f_1 \neq$ grad f_2 on U. Then the constant rank level set theorem yields that $\{f_1 = f_2\}$ is a regular submanifold of U of codimension 1, hence it is of measure zero. The proof is complete. $\square$

The following theorem is the main result of this section.

Theorem 4.15. *$\mu(x_1, \ldots, x_N)$ has a unique Fréchet median for almost every $(x_1, \ldots, x_N) \in M^N$.*

Proof. Since $M^N \setminus \bigcup_{k=1}^6 C_k$ is second countable, it suffices to show that for every $(x_1, \ldots, x_N) \in M^N \setminus \bigcup_{k=1}^6 C_k$, there exists $\delta > 0$ such that $\mu(y_1, \ldots, y_N)$ has a unique Fréchet median for almost every $(y_1, \ldots, y_N) \in B(x_1, \delta) \times \cdots \times B(x_N, \delta)$. In fact, let $(x_1, \ldots, x_N) \in M^N \setminus \bigcup_{k=1}^6 C_k$, without loss of generality, we can assume that $Q(x_1, \ldots, x_N) = \{y, z, x_N\}$, where $y, z \notin \{x_1, \ldots, x_N\}$. Assume that $Y = (y, n_1, \ldots, n_N, r_1, \ldots, r_N)$ and $Z = (z, v_1, \ldots, v_N, t_1, \ldots, t_N) \in \hat{\varphi}^{-1}(x_1, \ldots, x_N)$. Since $(x_1, \ldots, x_N)$ is a regular value of $\hat{\varphi}$, we can choose a $\delta > 0$ such that there exist neighborhoods U_1 of Y and U_2 of Z such that for $i = 1, 2$, $\hat{\varphi}_i = \hat{\varphi}|_{U_i} : U_i \to U$ is diffeomorphism, $\sigma(U_1) \cap \sigma(U_2) = \phi$ and $B(x_N, \delta) \cap (\sigma(U_1) \cup \sigma(U_2)) = \phi$, where $U = B(x_1, \delta) \times \cdots \times B(x_N, \delta)$. Furthermore, by Theorem 2.3 and Lemma 4.12, we can also assume that for every $(y_1, \ldots, y_N) \in B(x_1, \delta) \times \cdots \times B(x_N, \delta)$, $Q(y_1, \ldots, y_N) \subset B(x_N, \delta) \cup \sigma(U_1) \cup \sigma(U_2)$ and $Q(y_1, \ldots, y_N) \cap B(x_N, \delta) \subset \{y_N\}$. Now it suffices to use Lemma 4.14 to complete the proof. $\square$

Remark 4.16. In probability language, Theorem 4.15 is equivalent to say that if $(X_1, \ldots, X_N)$ is an M^N-valued random variable with density, then $\mu(X_1, \ldots, X_N)$ has a unique Fréchet median almost surely. Clearly, the same statement is also true if $X_1, \ldots, X_N$ are independent and M-valued random variables with desity.

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